Acta Mathematica Vietnamica
https://doi.org/10.1007/s40306-024-00526-8

Uniform Symbolic Topologies and Hypersurfaces

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This paper is dedicated to Ngô Viêt Trung whose transformative work in commutative algebra has helped to shape the field for decades, and who has constantly encouraged and supported younger generations

Received: 5 October 2023 / Accepted: 26 December 2023

Abstract
We study the question of which rings, and which families of ideals, have uniform symbolic topologies. In particular, we show that the uniform symbolic topology property holds for all dimension one primes in any normal complete local domain, provided dimension one primes in hypersurfaces have the uniform symbolic topology property. We also discuss bootstrapping techniques and provide a strong bootstrapping statement in positive characteristic. We apply these techniques to give families of primes in hypersurfaces of positive characteristic which have uniform symbolic topologies.

Keywords Symbolic powers · Uniform Artin-Rees

Mathematics Subject Classification (2010) 13A02 · 13F20 · 13H15

1 Introduction

In this paper we are interested in the following question.

Question 1.1 *Let R be a normal, excellent local domain and $X \subseteq Spec(R)$. Does there exist a positive integer b such that for all prime ideals $P \in X$, $P^{(bn)} \subseteq P^n$ for all $n \geq 1$?*[1]

Here we write $P^{(t)}$ to denote the t^{th} symbolic power of the prime ideal P, namely $P^{(t)} = P^t R_P \cap R$. For any Noetherian ring R, when b as above exists, we shall say that X satisfies

[1] In our previous papers [22–24] this question was formulated for complete local domains and $X = \mathrm{Spec}(R)$, but this turns out to be false in the absence of normality. See [25].

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Published online: 26 March 2024

the *uniform symbolic topology property* on prime ideals. Of particular interest is the case $X = \mathrm{Spec}(R)$, in which case we say that R satisfies the uniform symbolic topology property. Uniform results of this type for regular rings were first given by Ein, Lazarsfeld and Smith in [10], by Hochster and Huneke in [14], and recently by Ma and Schwede in mixed characteristic [30]. In these papers, the authors prove that if R is a regular local ring and d is the Krull dimension of R, then $P^{((d-1)n)} \subseteq P^n$, for all prime ideals $P \subseteq R$ and all $n \geq 1$. In [23], uniform results were proved for isolated singularities, under some mild conditions on the ring. Because a complete local domain containing a field, or an affine domain over a field, is a finite extension of a finite dimensional regular domain containing a field, it is natural to consider how the uniform symbolic topology property behaves with respect to finite ring extensions. Thus, Question 1.1 would have a positive answer for such rings if whenever $S \subseteq R$ is a finite extension of Noetherian domains, R has the uniform symbolic topology property on prime ideals if S has the uniform symbolic topology property on prime ideals. In [24] ascent and descent theorems of this type were proved (see [24, Corollaries 4.5 and 3.4]). Although descent of the uniform symbolic topology property holds, the results in [24] for ascent are not strong enough to give a positive answer to Question 1.1 outright. On the other hand, the ascent and descent results from [24] play an important role in our recent paper [22], where we show that (modulo some mild hypotheses), the uniform symbolic topology property holds in abelian extensions of regular rings.

In this paper, we use these ascent and descent theorems to make a reduction of the question above to hypersurfaces when X is the set of dimension one primes. In particular, using elementary Galois theory, we show that the family of dimension one primes in any complete normal local domain has the uniform symbolic topology property, if in all hypersurfaces, the family of dimension one, *regular* primes has the uniform symbolic topology property. However, the hypersurface case itself appears rather difficult, even in the case where the ring R is obtained by adjoining the n^{th} root of a square-free element of the base ring (see [22, Theorem 4.4]).

A key ingredient in [14, 15] and [23], is the existence of ring elements that uniformly multiply large symbolic powers of a family of ideals into smaller powers of the given ideals. In [22], we formalized this by calling such elements *uniform symbolic multipliers*. In section four, we prove the existence of uniform symbolic multipliers for hypersurfaces defined by a separable polynomial. In section five, we provide bootstrapping results for hypersurfaces. By bootstrapping, we mean results that guarantee the uniform symbolic topology property holds if for some c, k fixed, $P^{(c)} \subseteq P^k$, for all prime ideals P. Aside from their intrinsic interest, boostrapping results enable us to strengthen the results in section three regarding reduction of the general problem to the case of hypersurfaces. In many cases, finding a uniform k such that $P^{(k)} \subseteq P^2$, for all $P \in X$, suffices for the uniform symbolic topology property. Results of this type appear in section five. Finally, we use the techniques developed in section five to prove that certain families of regular primes satisfy the uniform symbolic topology property in many local hypersurfaces of positive characteristic.

For unexplained terminology, we refer the reader to the book [8]. Preliminary results and basic definitions are contained in Section 2. For a more detailed history of the problem at hand, we refer the reader to [23] or [24] and for unexplained terminology, we refer the reader to the book [8].

2 Preliminaries

In this brief section we record the results that we will rely upon throughout the paper. Our work relies heavily on both the Uniform Artin-Rees Property and the Uniform Briançon-Skoda Property. Because of this dependence, many of our theorems need to assume we are in a position to use them. This leads to the following definition:

Definition 2.1 Throughout this paper, we say that a reduced Noetherian ring S satisfies our *standard hypothesis* if for every finite extension T of S and reduced ideal $J \subseteq T$, T/J satisfies both the Uniform Artin-Rees Property and the Uniform Briançon-Skoda Property.

For the reader's convenience, we recall the definitions:

Definition 2.2 Let S be a Noetherian ring. We say that S satisfies the *Uniform Artin-Rees Property* if for all finitely generated S-modules $N \subseteq M$, there exists an integer k (depending on N and M) such that for all ideals I of S, and for all $n \geq k$

$$I^n M \cap N \subseteq I^{n-k} N.$$

Definition 2.3 Let S be a Noetherian reduced ring. We say that the *Uniform Briançon-Skoda Property* holds if there exists a positive integer k such that for all ideals I of S, and for all $n \geq k$,

$$\overline{I^n} \subseteq I^{n-k}.$$

Here we are writing $\overline{J}$ to denote the integral closure of an ideal J.

Let S be a reduced Noetherian ring. By [19, Theorems 4.12 and 4.13], in each of the following cases, S satisfies our standard hypothesis.

i) S is essentially of finite type over an excellent Noetherian local ring.
ii) S is a ring of characteristic p, and under the Frobenius map $F : S \to S$, S is a finite module over the image of the Frobenius map. If S is reduced, this is equivalent to saying that $S^{1/p}$ is module finite over S.
iii) S is essentially of finite type over $\mathbf{Z}$.

Two main results of [24], which we use freely in this paper, are the ascent and descent theorems mentioned in the introduction. Note that in [24], the base ring S is assumed to be *acceptable*, meaning it satisfies one of the three conditions above. In fact, the results below from [24] hold when S satisfies our standard hypothesis, since in [24] we used the acceptable hypothesis in order to invoke the Uniform Artin-Rees and the Uniform Briançon-Skoda properties.

Theorem 2.4 *(Ascent) Let $S \subset R$ be a finite integral extension of Noetherian domains such that S satisfies our standard hypothesis. Assume further that S is integrally closed and the quotient field of R is separable over the quotient field of S. If S has the uniform symbolic topology property on prime ideals, then R has the uniform symbolic topology property for all prime ideals $Q \subseteq R$ such that Q is the only prime lying over $Q \cap S$. Moreover, if R is also integrally closed, the conclusion holds for an arbitrary extension of quotient fields.*

Theorem 2.5 *(Descent) Let $S \subset R$ be a finite integral extension of Noetherian domains. Assume that S satisfies our standard hypothesis and is integrally closed. There exists an integer r, depending only on the extension $S \subset R$, such that if Q is a prime in R, $q = S \cap Q$, and $Q^{(bn)} \subset Q^n$, for some fixed b and for all $n \geq 1$, then $q^{(rbn)} \subset q^n$ for all $n \geq 1$. In particular, if R satisfies the uniform symbolic topology property, then so does S.*

As a consequence of the descent theorem, for example, if a finite group acts on a regular ring which satisfies our standard hypothesis and is of equicharacteristic zero having finite Krull dimension, then the ring of invariants must satisfy the uniform symbolic topology property on prime ideals.

3 Reduction to Hypersurfaces

The main theorem of this section is a reduction theorem which proves that the property of a ring having uniform symbolic topology property can often be reduced to a special class of prime ideals in hypersurface rings. The main techniques used are from Galois theory and the uniform Artin-Rees theorem. We begin with three propositions concerning contractions of prime ideals inside subrings of Galois extensions. Our main result is Theorem 3.6 below.

Proposition 3.1 *Let $S \subset R$ be a finite Galois extension of normal Noetherian domains. Let $Q \subset R$ be a prime ideal. Then there exist an element $v \in R$ and a normal domain T with $S[v] \subset T \subset R$ such that*

(a) *$T \subset R$ is a Galois extension and Q is the only prime in R lying over $Q \cap T$.*
(b) *For $B := S[v]$, B and T have the same fraction field and $Q \cap T$ does not contain the conductor of T into B. In particular, $Q \cap T$ does not contain $f'(v)$, where $f(x)$ is the minimal polynomial of v over the quotient field of S.*

Proof Let K denote the fraction field of S and L denote the fraction field of R. Let G be the Galois group of L over K and set $q = Q \cap S$. Let $Q = Q_1, Q_2, \dots, Q_r$ denote the prime ideals in R lying over q. They are exactly the conjugates of Q under the action of G.

Let H be the decomposition group of Q, that is, the subgroup of automorphisms $g \in G$ such that $g(Q) = Q$. Let E be the fixed field of H, and let T be the integral closure of S in E. It is well known that Q is the unique prime ideal of R lying over $Q \cap T$, which gives the first statement of the theorem, as R is Galois over T by construction. For example, see [5]. However, we need some notation to exhibit this explicitly in order to prove part (b). For $2 \leq i \leq r$, choose an element $g_i \in G$ such that $g_i(Q) = Q_i$. We set $g_1 = 1$, the identity. Then

$$G = H \cup g_1 H \cup \ldots \cup g_r H$$

is a coset decomposition of G over H. Note that the degree of the extension $K \subset E$ is r.

To prove the second statement, choose $u \in Q$, $u \notin \bigcup_{2 \leq i \leq r} Q_i$. For $g \in H$, observe that $g(u) \notin Q_j$ for every $2 \leq j \leq r$. For if $g(u) \in Q_j$, then $u \in g^{-1}(Q_j)$, which by the choice of u means that $g^{-1}(Q_j) = Q$, or $g(Q) = Q_j$, a contradiction. Set $v = \prod_{g \in H} g(u)$. Clearly v is invariant under H, and is therefore in T. Moreover, $v \in Q$ and $v \notin \bigcup_{2 \leq i \leq r} Q_i$. We claim that v is an element which satisfies the second condition of the theorem.

We first prove that $B := S[v]$ has the same fraction field as T. To prove this it suffices to see that the degree of B over S is the same as that of T over S, which is r. We prove this by noting that v has r distinct conjugates, namely $v, g_2(v), \dots, g_r(v)$ which must be distinct since if $h(v) = v$, then h must be in H, as $v \in Q$ and $v \notin \bigcup_{2 \leq i \leq r} Q_i$. It follows that the minimal polynomial of v over K is exactly $\prod_{1 \leq i \leq r}(X - g_i(v))$.

It remains to prove that $Q \cap T$ does not contain the conductor of T into B. Since T is the integral closure of B, it is well known that the element $f'(v)$ is in the conductor of T into B (see, for example, [43, Theorem 12.1.1]). We have $f'(v) = (v - g_2(v)) \cdots (v - g_r(v))$, so that as a polynomial expression in v, every term in $f'(v)$ is a multiple of v, and hence in

Q, except the constant term which up, to a sign, is $g_2(v)\cdots g_r(v)$, which is not in Q. Thus, $f'(v) \notin Q \cap T$. □

We record the following corollary of Proposition 3.1 which is interesting in its own right.

Corollary 3.2 *Let S be a regular local ring and R the integral closure of S in a finite Galois extension of its quotient field. If $Q \subseteq R$ is a prime ideal, then there exists a normal local domain $S \subseteq T \subseteq R$ such that Q is the only prime in R lying over $Q \cap T$ and $T_{Q\cap T}$ is a regular local ring.*

Proof Let $Q \subseteq R$ be a prime ideal. The proof of Proposition 3.1 shows that there exist a normal domain $S \subseteq T \subseteq R$ and $v \in T$ such that $S[v]$ and T are birational, Q is the only prime lying over $Q\cap T$ and $f'(v) \notin Q\cap T$, where $f(x)$ is the minimal polynomial of v over the quotient field of S. Since S is regular, $S[v]_{Q\cap S[v]}$ is regular, since $f'(v) \notin Q \cap S[v]$. On the other hand, $f'(v)$ is in the conductor of T into $S[v]$, so $T_{Q\cap T} = S[v]_{Q\cap S[v]}$ is regular, as required. □

Proposition 3.3 *Let $A \subset B$ be Noetherian integral domains having the same fraction field. Assume that A satisfies the uniform symbolic topology property, B is a finite extension of A, and B has the uniform Artin-Rees property. Then there exists an integer e such that for all primes Q in B such that $Q \cap A$ does not contain the conductor of B into A and for all $n \geq 1$, $Q^{(en)} \subset Q^n$.*

Proof Fix an integer c such that $q^{(cn)} \subset q^n$ for all primes q of A and for all $n \geq 1$. Let t be any element of the conductor and choose $f \geq 1$ such that f is a uniform Artin-Rees number for the module $tB \subset B$. We claim that $e := c(f+1)$ satisfies the conclusion of the proposition. Let Q be a prime in B not containing the conductor, set $q := Q \cap A$ and let $u \in Q^{(en)}$. We have that $u \in Q^{en}B_Q \cap B = q^{en}A_q \cap B$, since Q does not contain the conductor. It follows that $tu \in q^{en}A_q \cap A = q^{(en)} \subset q^{(f+1)n}$. Hence

$$tu \in Q^{(f+1)n} \cap tB \subset tQ^{(f+1)n-f}.$$

We cancel t to obtain that $u \in Q^{(f+1)n-f} \subset Q^n$, as desired. □

Proposition 3.4 *Let $S \subseteq R$ be a finite Galois extension of normal Noetherian domains. Assume S satisfies our standard hypothesis. Then there exist finitely many simple, integral extensions $S[v_i] \subseteq R$, $1 \leq i \leq t$, with the following property: If each $S[v_i]$ satisfies the uniform symbolic topology property for each prime ideal not containing the conductor of $S[v_i]$ into its integral closure, then R satisfies the uniform symbolic power property.*

Proof Let K denote the quotient field of S and L denote the quotient field of R. Since L is Galois over K, there are only finitely many intermediate fields $K \subseteq E \subseteq L$. Thus, there are only finitely many normal domains $S \subseteq T \subseteq R$, namely, the integral closure of S in each intermediate field E. The idea is to now apply the previous two propositions.

Fix one such T and define Λ_T to be the set of prime ideals Q of T with the property that there exists an element $v_Q \in T$ such that $S[v_Q]$ is birational to T and Q does not contain $f'_{v_Q}(v_Q)$, where $f_{v_Q}(X)$ is the minimal polynomial for v_Q over K. If this set is nonempty, let I_T be the ideal generated by all of the $f'_{v_Q}(v)$ obtained for each $Q \in \Lambda_T$. Then I_T is not contained in Q for every $Q \in \Lambda_T$, and moreover, I_T is generated by finitely many $f'_{v_1}(v_1), \ldots, f'_{v_r}(v_r)$ corresponding to $S[v_i] \subset T$, $1 \leq i \leq r$. Since T inherits the uniform Artin-Rees theorem from S (by [19]), it follows from Proposition 3.3 that if $S[v_i]$ satisfies the uniform symbolic topology property at primes not containing the conductor, there is an integer e_i such that for all $n \geq 1$ and every $Q \in \mathrm{Spec}(T)$ such that Q does not contain $f_i'(v_i)$, $Q^{(e_i n)} \subseteq Q^n$, since $f_i'(v_i)$ is in the conductor of T into $S[v_i]$. By choosing e_T to be the maxmium of all e_i, we obtain that for all $Q \in \Lambda_T$, and for all $n \geq 1$, $Q^{(e_T n)} \subset Q^n$.

Now, for each T we obtain finitely many rings of the form $S[v]$ with the property that if each $S[v]$ satisfies the uniform symbolic topology property, then there exists an e_T as in the previous paragraph, a uniform constant for Λ_T. For the required collection $S[v_i]$, we take the finite collection of rings of the form $S[v]$ obtained from each T as above and take e to be the maximum e_T as T ranges over the finitely many normal domains between S and R. Thus, $Q^{(en)} \subseteq Q^n$, for all n and all $Q \in \Lambda_T$, for all T.

Finally, let C_T denote the prime ideals P in R such that $P \cap T \in \Lambda_T$. By Proposition 3.1 every prime in R belongs to some C_T. By the previous paragraph and the ascent property (Theorem 2.4), it follows that for each T, there is an integer k_T, depending only on e and T such that $P^{(k_T n)} \subset P^n$ for all $P \in C_T$. Taking k to be the maximum of the k_T shows that R satisfies the uniform symbolic topology property. □

Remark 3.5 While the proposition above is accurate, as of now it has limited applicability. This is because, in an arbitrary hypersurface A, there may exists prime ideals $P \subseteq A$ so that the symbolic and P-adic topolgies are not equivalent - and thus the uniform symbolic topology property could not hold. For an example of this, see [25]. However, the reduction in the proposition above depends upon the choice of $u \in Q$, for the primes ideals $Q \subseteq R$. It is conceivable that a more careful choice of $u \in Q$ (or u in Q extended to some generic extension of R) has the property that the hypersurface $S[v]$ derived from u has the property that for any prime $P \subseteq S[v]$, the symbolic and P-adic topologies are equivalent. Such a choice is available in the example from [25]. However, for dimension one primes in R, this is no longer an issue, since, by Schenzel's criterion, any such prime in a complete local domain has the property that its adic and symbolic topologies are equivalent, so we state the following theorem for this class of primes.

We now combine the results of this section to prove the following theorem. If A is a Noetherian ring and $P \subseteq A$ is a prime ideal, we will say that P is a *regular* prime if A_P is a regular local ring. We say that A has the uniform symbolic property for its regular primes, if there exists $c > 0$ such that $P^{(cn)} \subseteq P^n$, for all regular primes P and all $n \geq 1$.

Theorem 3.6 *Consider the following statements.*

(a) *Complete local hypersurface rings have uniform symbolic topologies for their dimension one, regular primes.*
(b) *All normal complete local domains have uniform symbolic topologies at dimension one, regular primes.*

If either (a) *or* (b) *holds, then the family of dimension one primes in all normal complete local domains have uniform symbolic topologies.*

Proof Let R be a normal, complete local domain. Then R is finite over a complete regular local ring S. Note that S satisfies our standard hypothesis. Let K denote the quotient field of S and L the quotient field of R. We want to reduce to the case that R is a Galois extension of S. By taking the integral closure of R in the normal closure of L, we obtain a normal, complete local domain R' containing R such that $S \subseteq R'$ is a finite extension of normal domains. By the descent property Theorem 2.5, the uniform symbolic topology property descends in a finite extension of normal domains (for which Uniform Artin-Rees holds), so that if R' has the uniform symbolic topology property, then R has the uniform symbolic topology property. After changing notation, it now follows that, for either statement (a) or (b), we may begin again assuming that L is a normal extension of K.

If S has characteristic zero, then R is a Galois extension of S. If S has characteristic p, let E be the separable closure of K in L. Then $K \subseteq E$ is a Galois extension and L is purely inseparable over E. Let R' denote the integral closure of S in E, so that $R' \subseteq R$ is a finite extension of complete, normal, local domains. Note that each prime ideal Q in R lies over exactly one prime in R'. Thus, by the ascent property, Theorem 2.4, R has the uniform symbolic topology property, if R' has the uniform symbolic topology property. It now follows that to prove the theorem, we may assume that R is a Galois extension of S.

For statement (a), by Proposition 3.4 (and its proof), there exist finitely many simple extensions $S[v_i] \subseteq R$ with the property that if each $S[v_i]$ satisfies the uniform symbolic topology property at each prime not containing $f_i'(v_i)$, then R satisfies the uniform symbolic topology property. Here $f_i(x)$ is the minimal polynomial of v_i over K. However, for any prime $P \subseteq S[v_i]$ not containing $f_i'(v_i)$, $S[v_i]_P$ is regular. Hence by our assumption, $S[v_i]$ satisfies the uniform symbolic topology property for all primes not containing $f_i'(v_i)$, which gives what we want.

For statement (b), let $T_1, \ldots, T_s$ be the set of normal domains between S and R. Note that each T_j is a complete normal local domain. By Corollary 3.2, for each prime ideal $Q \subseteq R$, there is a T_i such that $Q \cap T_i$ is a regular prime and Q is the only prime lying over $Q \cap T_i$. Thus, if all of the T_i have uniform symbolic topologies at regular primes, then as in the proof of Proposition 3.4, the uniform symbolic topology property lifts to all primes in R, via Theorem 2.4. □

4 Uniform Symbolic Multipliers for Hypersurfaces

A key ingredient in a number of papers that deal with the uniform symbolic topology property is the existence of ring elements that multiply large symbolic powers of an ideal into powers of smaller symbolic powers. In [22], we formalized this notion with the following definition.

Definition 4.1 Let R be a Noetherian ring and U a set of ideals of R (for example, all prime ideals or all reduced ideals). We say that a non-zerodivisor $x \in R$ is a *uniform multiplier for symbolic powers with respect to* U with if there exists $k \geq 1$ such that for all ideals $I \in U$, $x^n I^{(kn+en)} \subseteq (I^{(e+1)})^n$ for all $e \geq 0$ and $n \geq 1$. If $U = \mathrm{Spec}(R)$, we just say that x is a uniform multiplier for symbolic powers. In either case, we refer to the integer k as the *index* of the multiplier x.

The purpose of this section is to prove the existence of uniform multipliers for symbolic powers over separable hypersurfaces. As a corollary we are able to strengthen the reduction to hypersurfaces theorem from the previous section (see Corollary 4.6 below).

Remark 4.2 It follows from [14, Theorem 1.1], that if R is a finite-dimensional regular domain containing a field, then $1 \in R$ is a uniform multiplier for symbolic powers for all ideals. The same theorem shows that if R is a geometrically reduced affine domain over a field K (which in the case that R has characteristic zero, just means that R is reduced), then any x in the square of the Jacobian ideal of R over K is a uniform multiplier for symbolic powers for all ideals. In [23, Proposition 3.4], it is shown that if R is a Noetherian domain containing a field of characteristic $p > 0$ such that R is F-finite and an isolated singularity, then there exists an $\mathfrak{m}$-primary ideal consisting uniform multipliers for symbolic powers. Additionally, in [22], we prove the existence of uniform multipliers for symbolic powers in certain types of repeated radical extensions.

The following crucial lemma appears as [22, Lemma 3.3], though it has a number of closely related antecedents (see [14] and [23], for example) .

Lemma 4.3 *Suppose that R is a d-dimensional F-finite integral domain containing a field of characteristic $p > 0$. Fix $a \in R$ and assume there are flat R-modules $F_q \subseteq R^{1/q}$ such that $aR^{1/q} \subseteq F_q$ for all q. Then for every ideal $I \subseteq R$, if we let h denote the maximum of the analytic spreads of the ideals I_P, where P is an associated prime of I, then $a^n I^{(nh+en)}$ is contained in the tight closure of $(I^{(e+1)})^n$ for all $e \geq 0$ and $n \geq 1$. Moreover, if a is also a test element for R, then a^2 is a uniform multiplier for symbolic powers, with index d, for all ideals of R.*

We now prove that if R is a hypersurface defined by the separable polynomial, $f(x)$, then $f'(x)^2$ is a uniform multiplier for symbolic powers.

Proposition 4.4 *Let S be an excellent regular domain containing a field and suppose R is a finite extension of the form $R = S[x]$, where x is the root of a monic separable polynomial $f(X)$ such that $R = S[X]/(f(X))$. We assume that R is reduced and that $f'(x)$ is not a zerodivisor in R. Further assume that if S has positive characteristic, then S is F-finite. Then $f'(x)^2$ is a uniform symbolic multipler with index $d := \dim(R)$ for the set of radical ideals in R.*

Proof We first consider the case that S contains a field of characteristic p. For this, we will first show that for every ideal $I \subseteq R$, $f'(x)^n I^{(nh+en)}$ is contained in the tight closure of $(I^{(e+1)})^n$ for all $e \geq 0$ and $n \geq 1$, where h denotes the maximum of the analytic spreads of the ideals I_P, where P is an associated prime of I. By the previous lemma, we must find flat R-modules F_q for which $f'(x)R^{1/q} \subseteq F_q$ for all q. We set $F_q := S^{1/q}[R] = S^{1/q}[x]$ for all q.

First note that since S is regular, $S^{1/q}$ is flat over S and since R is free over S, $S^{1/q} \otimes_S R$ is flat over R. On the other hand, as in the proof of [15, Lemma 3.4], $S^{1/q} \otimes_S R$ is isomorphic to $S^{1/q}[R] = S^{1/q}[x]$ (since, for example, $R_{f'(x)}$ is regular). Thus, we take $F_q := S^{1/q}[x]$ for each q.

We now note that $f'(x)R^{1/q} \subseteq F_q$ for all q. To see this, let K denote the total quotient ring of R, so that $K = K_1 \times \cdots \times K_r$, where each K_i is the quotient field of R/P_i, with $P_1, \ldots, P_r$ denoting the minimal primes of R. Note that in fact each P_i is a principal ideal generated by an irreducible factor, say $g_i(X)$, of $f(X)$. Let $R_i := R[X]/(g_i(X))$ and write x_i for the image of x in R_i, so that $R_i = S[x_i]$. Now K_i is separable over L (since the image $f'(x)$ in K_i is non-zero) and thus, $L^{1/q} \otimes_S K_i = K_i^{1/q}$, as $L^{1/q}$ is purely inseparable over L (by [26, Theorem 21]). It follows that for each i, $K_i^{1/q} = L^{1/q}[x_i]$, so that x_i is a primitive element for $K^{1/q}$ over $L^{1/q}$. Thus, $g_i'(x_i)R_i^{1/q} \subseteq S^{1/q}[x_i]$. Since $g_i'(x_i)$ divides $f'(x)$, $f'(x)R_i^{1/q} \subseteq S[x_i]$, for all q and all i. Since as subrings of $K^{1/q} = K_1^{1/q} \times \cdots \times K_r^{1/q}$, $R^{1/q}$ can be identified with $R_1^{1/q} \times \cdots \times R_r^{1/q}$ and $S^{1/q}[x]$ can be identitifed with $S^{1/q}[x_1] \times \cdots \times S^{1/q}[x_r]$, it follows that $f'(x)R^{1/q} \subseteq S^{1/q}[x]$, as required.

Now, by [13, Theorem 6.13], since $f'(x)R^{1/q} \subseteq S^{1/q}[R]$ for all q, $f'(x)$ is a test element for R and thus, $f'(x)^{2n} I^{(nh+en)} \subseteq (I^{(e+1)})^n$, as required.

If S contains a field of characteristic zero, the proof of the result we seek proceeds via reduction to characteristic p. The proof follows along the same lines as most reduction to characteristic p proofs. In particular, we can follow the ideas in the proof of [14, Theorems 4.3 and 4.4], and also the proof of Theorem 4.7 and in the Appendix in [21]. To elaborate, the results in [14], show how, starting with a complete local ring A, say, and a counter-example to an inclusion of the type we want involving symbolic powers, one can produce a counter-example in a ring of positive characteristic - the point being that conditions like elements belonging to, or not belonging to, various symbolic powers, as well as the maximum of local analytic spreads can be preserved via the reduction process. On the other hand, we need a slight variation of this, because we will be working with two rings at once, S and its simple extension R – but [21, Theorem 4.7 and the Appendix], illustrate how to carry the ring strucure of R along in the reduction process. Another crucial point here is that the failure of the required property of a proposed uniform multiplier for symbolic powers can be preserved along the way, because the element $f'(x)^2$ is given *a priori* as an element of the original ring – in this case R.

We now sketch out the steps required in order to reduce our statement to positive characteristic. So, we assume that we have $I \subseteq R$, $u \in I^{(hn+en)}$ with $f'(x)^{2n}u \notin (I^{(e+1)})^n$, for some $e \geq 0$, $n \geq 1$. By standard localization arguments, our counter-example persists after we localize at some prime ideal in S, so we may assume that we have a counter-example when S is a regular local ring. We now lift the counterexample by completing S at its maximal ideal. Writing $\hat{S}$ for the completion of S, we have $\hat{R} = \hat{S} \otimes_S R$, which is faithfully flat over R. Note also that since S is excellent, $\hat{R}$ remains reduced, as does $\hat{R}/I\hat{R} = \hat{R}/\hat{I}$. Morever, since $\hat{R}/I\hat{R}$ is faithfully flat over R/I, non-zerodivisors on the latter remain non-zerodivisors on the former, so that if $U \subseteq R$ denotes the set of non-zerodivisors on R/I and $W \subseteq \hat{R}$ denotes the set of non-zerodivisors on $\hat{R}/\hat{I}$, then $U = W \cap R$ and hence $(R/I^k)_U \hookrightarrow (\hat{R}/\hat{I}^k)_W$, so that $\hat{I}^{(k)} \cap R = I^{(k)}$, for all $k \geq 1$. Thus, $u \in \hat{I}^{(hn+en)}$. On the other hand, we clearly have $I^{(k)}\hat{R} = I^k\hat{R}_U \cap \hat{R} \subseteq I^k\hat{R}_W \cap \hat{R} = \hat{I}^{(k)}$. If P is an associated prime of $\hat{R}/\hat{I}^{(k)}$, $P \cap W = \emptyset$, so P is contained in a minimal prime of $\hat{I}$, and thus P is also an associated prime of $I^{(k)}\hat{R}$. It follows from this that $I^{(k)}\hat{R} = \hat{I}^{(k)}$, for all k. Therefore, $((I\hat{R})^{(e+1)})^n = (\hat{I}^{(e+1)})^n$, and therefore, $a^n u \notin (\hat{I}^{(e+1)})^n$. So, we may begin again assuming that S is a complete regular local ring, and we have a counter-example in R as above to our proposition.

At this point one uses Artin approximation to find an counter-example in an affine algebra over a field of characteristic zero. If we were only working with S, then by [14, Theorem 4.3], we could create a counter-example in an affine algebra, but we need to preserve our counter-example in a ring over S. We may therefore, follow the path laid out in [21, Theorem 4.7 and the Appendix]. One uses equations over S to capture the ring structure of R. For example, since R is free over S with basis $1, x, \dots, x^{n-1}$, where $f(X)$ has degree n, one writes each product $x^i \cdot x^j$ in terms of the basis with coefficients in S. The resulting equations can be thought of solutions over S to a system of equations in n variables over S. Similarly, one can realize the associative property of multiplication and the distributive property as solutions to equations over S. Since the ideal I is a submodule of R as an S-module, one can choose a set of generators for I and write equations expressing the closure of I under multiplication by elements of R, using the consequences of taking products of the basis elements of R over S with the generators of I as an S-module. As in [14, Theorem 4.3], one can transfer all of this data and the attendant data associated to our counter-example to a finitely generated algebra over the coefficient field, say E, of S. Here we are thinking of S as a formal power series ring in d variables over E. In fact, one first adjoins to E all of the relevant elements from S that are solutions to the various equations tracking the data to obtain a subring ring S_0 and then uses [14] (which relies upon [1]) to find a ring S_1 and maps $S_0 \hookrightarrow S_1 \to S$ in which all of the conditions from S are preserved, and such that all of the ideals and modules that we started with in S are obtained by tensoring their counterparts in S_1 with S over S_1. Moreover,

S_1 is a regular ring and the counter-example in question holds in the extension $R_1 := S_1[x]$. Note that this can be done so that the element x still satisfies the equation $f(x) = 0$, $f'(x)$ is a non-zerodivisor in R_1 and the rings R_1 and R_1/I_1 are reduced, where I_1 is generated by the images in S_1 of the original generators of I. Now, strictly speaking, the field E is not the original field E_0 contained in the original S, but one can assume $E_0 \subseteq E$, and the last paragraph of [16, Theorem 3.5.1] explains how to reduce to the case that $E_0 = E$.

The next step is to reduce to an affine algebra over $\mathbb{Z}$, which can be done in a standard way by collecting all coefficients of all the finitely many equations which describe our situation, and then letting A be the finitely generated $\mathbb{Z}$-algebra obtained as the subring of the base field given by adjoining those finitely many elements to $\mathbb{Z}$. One further uses generic flatness to insure that after creating models R_A and $(R/I)_A$ of R and R/I over this finitely generated $\mathbb{Z}$-subalgebra A of k, there exists a dense subset of closed points $S \subseteq \operatorname{Spec} A$ such that we still a counterexample after moding out any one of the closed points in S. These counterexamples now live in positive characteristic, and by choosing the characteristic large enough we can avoid any divisors of our the degree of $f(X)$, so that $f(X)$ remains separable. Moreover, as described in [16, Chapter 2] and [14, Theorem 4.3], we retain all relevant information, including various ideals being reduced, and analytic spreads. This leads to a contradiction to the positive characteristic case. □

The proposition above enables us to extend the main theorem from the previous section. For this, we must use the following bootstrapping theorem from [22].

Theorem 4.5 *Let R be a Noetherian ring. Let U be a set of ideals of R, and suppose $x \in R$ is a uniform symbolic multiplier with index $k \geq 1$ for the set U. Assume further that the pair $(x) \subseteq R$ has uniform Artin-Rees number $l \geq 1$. If there exists $b \geq 1$ such that $I^{(b+1)} \subseteq I^{l+1}$, for all ideals $I \in U$, then for $d = k + b$, $I^{(dn)} \subseteq I^n$, for all $n \geq 1$ and all $I \in U$.*

Here is the extension of Theorem 3.6.

Corollary 4.6 *Let $\mathcal{C}$ denote the class of hypersurfaces T of the following type: $T = S[X]/(f(X))$, where S is a complete regular local ring, $f(X)$ is a separable polynomial, and S is F-finite if S has positive characteristic. Suppose each T in $\mathcal{C}$ has the property that if l denotes the uniform Artin-Rees number for $(f'(x)) \subseteq T$, then there exists $k \geq 1$, such that $P^{(k)} \subseteq P^l$, for all dimension one regular primes $P \subseteq T$. Then the following statements hold:*

(a) *Every T in $\mathcal{C}$ satisfies the uniform symbolic topology property on the set of dimension one regular primes.*
(b) *In all normal, complete local domains, the family of dimension one primes satisfies the uniform symbolic topology property.*

Proof The first statement follows immediately from Proposition 4.4 and Theorem 4.5. The second statement follows from the proof of Theorem 3.6 (where the general case of a normal complete local domain reduces to the case of a Galois extension over a complete regular local ring), together with Proposition 4.4 and Theorem 4.5. □

5 Bootstrapping in Characteristic *p*

The purpose of this section is to prove a bootstrapping theorem for hypersurfaces in characteristic $p > 0$. Here we assume outright that our ring R is a hypersurface, and we consider the set $E \subseteq \mathrm{Spec}(R)$ of primes in the ring whose adic and symbolic topologies are equivalent - the largest possible set of primes to which a uniform statement could apply. In particular, we prove that if R is a complete local domain which is a hypersurface and has positive characteristic, then if we can find a uniform k such that for all primes $P \in E$, $P^{(k)} \subset P^2$, then E has the uniform symbolic topology property. Note that the power 2 on each P is now independent of the hypersurface, contrary to the statement of Corollary 4.6. Note also that this result applies to abstract hypersurfaces, not just those defined by a single separable polynomial over a regular local ring.

Throughout this section, if $R = S/I$ is a quotient of a regular local ring and P is a prime ideal of R, then by L_k we denote the unmixed part of $P^k + I$ in S, i.e., the lift of the kth symbolic power of P to S.

We need a lemma which is implicit in the work of [14], but which we state formally:

Lemma 5.1 *Let R be a Noetherian local ring of characteristic p and dimension d with infinite residue field. Let J be an ideal of R. Then for all $q = p^e$, $J^{qd} \subset (J^{[q]})^*$, the tight closure of the qth Frobenius power of J.*

Proof We can reduce modulo minimal primes to prove this assertion. See [14]. Henceforth we assume that R is a domain. Let K be a minimal reduction of J. Note that K is generated by at most d elements. Choose an element $c \neq 0$ such that $cJ^n \subset K^n$ for all $n \geq 1$ (see [43]). Let q' be a varying power of p, and let $u \in J^{qd}$. Then $cu^{q'} \in cJ^{qq'd} \subset K^{qq'd} \subset K^{[qq']} = (K^{[q]})^{[q']}$ implies that $u \in (K^{[q]})^* \subset (J^{[q]})^*$. □

Lemma 5.2 *Suppose R is a local domain containing an infinite field of characteristic $p > 0$ and $c \in R$ is such that R_c is regular. Assume $d = \dim(R)$. Let P be a prime ideal not containing c. Given $n = qd$ and $k \geq 1$, there exists a power c^N of c (depending on n, k and P), such that $c^N \cdot P^{(nd+nk)} \subseteq (P^{(k)})^{[q]}$.*

Proof Since R_c is regular, $P_c^{(nd+nk)} \subseteq (P^{(k)})_c^n$, so there exists $r > 0$ such that $c^r \cdot P^{(nd+nk)} \subseteq (P^{(k)})^n$. Taking $J = P^{(k)}$ in Lemma 5.1 we have $(P^{(k)})^{qd} \subseteq ((P^{(k)})^{[q]})^*$ and therefore $c^r \cdot P^{(nd+nk)} \subseteq ((P^{(k)})^{[q]})^*$. In R_c, $(P^{(k)})^{[q]})_c$ is tightly closed, and therefore, $c^t \cdot (c^r \cdot P^{(nd+nk)}) \subseteq (P^{(k)})^{[q]}$ for some $t > 0$. Taking $N = r + t$, gives what we want. □

Here is the key result leading to the bootstrapping theorem in characteristic $p > 0$.

Proposition 5.3 *Let R be a local domain of dimension d which is a hypersurface and has positive characteristic. Write $R = S/(f)$, where S is a regular local ring of characteristic $p > 0$. Let $P \in \mathrm{Spec}(R)$ such that R_P is regular. Denote by Q the lifting of P to S. Assume there exists an integer k and an ideal $J \subset Q$ such that $L_k \subset (J, f)$, and $J : f \subset Q$. Set $N = d^2 + kd$. Then $P^{(Nq)} \subset P^{[q]}$ for all $q = p^e$. Moreover, if we fix an integer n and write $q = an + r$, where $q = p^e$ is a varying power of p, and $0 \leq r \leq n - 1$, then $P^{Nr}(P^{(Nn)})^a \subset (P^n)^a$. In particular, $P^{(Nn)}$ is contained in the integral closure of P^n.*

Proof Temporarily fixing $n = qd$ and k, for $c \in R$ such that R_c is regular, Lemma 5.2 implies that some power of c multiplies $P^{(dn+kn)}$ into $(P^{(k)})^{[q]}$. Lifting back to S, and continuing to call the element c by the same name, we obtain that some power of c multiplies L_{qd^2+qkd} into $(L_k^{[q]}, f)$. Further multiplying by f^{q-1} yields that some power of c multiplies $f^{q-1}L_{qd^2+qkd}$ into $L_k^{[q]}$. By the construction of L_k, c is a non-zerodivisor modulo L_k, and then the regularity of S (i.e., flatness of the Frobenius) proves that c is a non-zerodivisor modulo $L_k^{[q]}$. It follows that $f^{q-1}L_{qd^2+qkd} \subset L_k^{[q]}$, independently of q and k, and therefore

$$L_{qd^2+qkd} \subset L_k^{[q]} : f^{q-1}.$$

By assumption, $L_k \subset (f, J)$. Hence

$$L_k^{[q]} : f^{q-1} \subset (f^q, J^{[q]}) : f^{q-1} = (f)+J^{[q]} : f^{q-1} \subset (f)+J^{[q]} : f^q = (f)+(J : f)^{[q]},$$

where the last equality holds because S is regular. By assumption, $J : f \subset Q$. Putting together all of these containments yields that

$$L_{qd^2+qkd} \subset L_k^{[q]} : f^{q-1} \subset (f) + Q^{[q]}.$$

Going modulo (f), we see that $P^{(Nq)} \subset P^{[q]}$ for all $q = p^e$, as claimed.

To finish the proof, we modify an argument in [14]. Let $u \in P^{(Nn)}$. For every $q = p^e$ we write $q = an + r$ where $a \geq 0$ and $0 \leq r \leq n-1$ are integers. Then $u^a \in P^{(Nan)}$ and $P^{Nr}u^a \subset P^{(Nan+Nr)} = P^{(Nq)}$. By the paragraph above, it follows that

$$P^{Nr}u^a \subset P^{[q]} \subset P^q \subset (P^n)^a,$$

which proves our claim. To prove the last statement, choose a nonzero element $d \in P^{Nr}$. For infinitely many values of a, we have that $du^a \in (P^n)^a$. Since R is a domain, it follows from [43, Cor. 6.8.12], that u is in the integral closure of P^n. □

The preceding work leads immediately to the main theorem of this section. For this we let E denote the prime ideals in Spec(R) such that the P-adic and P-symbolic topologies are equivalent. The first statement in the theorem below gives a surprising necessary condition for the P-adic and P-symbolic topologies to be equivalent.

Theorem 5.4 *Let R be an F-finite local domain of characteristic $p > 0$ and dimension d, which is a hypersurface. Let $P \subseteq R$ be a regular prime and assume there exists an integer k such that $P^{(k)} \subseteq P^2$. Then, there exists $c \geq 1$ such that $P^{(cn)} \subseteq P^n$ for all $n \geq 1$. In particular, if there exists $k \geq 1$ such that $P^{(k)} \subseteq P^2$, for all regular primes in the set E, then the set of regular primes in E satisfy the uniform symbolic topology property.*

Proof Maintaining the notation from the previous proposition, we claim that we may choose $J = Q^2$ in Proposition 5.3. Our assumption shows that $L_k \subset J + (f)$, so we need only to prove that $Q^2 : f \subset Q$. This is true since $f \notin Q^{(2)}$ since $(S/(f))_Q$ is a regular ring by assumption.

It follows from Proposition 5.3 that $P^{(Nn)}$ is contained the intergal closure of P^n, for all regular primes P and $n \geq 1$, where $N = d^2 + kd$. By the Uniform Briançon-Skoda Theorem (see [19]), there exists $t \geq 1$ such that the integral closure of P^n is contained in P^{n-t}, for all $n > t$, and all P. An elementary induction argument now shows that for $c := N\lceil \frac{t+2}{2} \rceil$, $P^{(cn)} \subseteq P^n$, for all $n \geq 1$ and all regular primes P, which is what we want. □

Corollary 5.5 *Assume that every F-finite characteristic p complete hypersurface satisfies the following property: There exists $k > 0$ (depending upon the ring) such that $P^{(k)} \subseteq P^2$ for all dimension one regular prime ideals P. Then the set of dimension one primes in every F-finite integrally closed complete local domain satisfies the uniform symbolic topology property.*

Proof Immediate from Theorems 3.6 and 5.4. □

We close this section by using the bootstrapping technique to provide a large classes of prime ideals in hypersurface rings have the uniform symbolic topology property. We let S be an F-finite regular local ring of prime characteristic, and let $R = S/(f)$ be a hypersurface. We define Λ to be the set of prime ideals q in R such that R/q is normal, and such that for every prime P containing q with $\dim(R/q)_P = 1$, R_P is regular. We observe that if the height of the ideal defining the singular locus of R is m, then every prime q in $\mathrm{Spec}(R)$ having height at most $m-2$, and such that R/q is normal, lies in Λ. This follows since every prime P with $\dim(R/q)_P = 1$ is necessarily of height at most $m-1$ in R, and therefore R_P will be regular.

Theorem 5.6 *Let S be an F-finite regular local ring of prime characteristic, and let $R = S/(f)$ be a hypersurface domain. Then the set of prime ideals in Λ have the uniform symbolic topology property. Namely there is an integer k such that for all $q \in \Lambda$ and for all $n \geq 1$,*

$$q^{(kn)} \subset q^n.$$

Proof For $q \in \Lambda$, let L_k be as above, namely the unmixed part in S of $(f) + Q^k$, where Q is the lifting of q to S. We wish to apply Proposition 5.3 with $J = Q^{(2)}$. By assumption, $R_Q = R_q$ must be regular. It follows that $f \notin Q^{(2)}$ and therefore $Q^{(2)} : f = Q$. Thus, if we can prove there is a uniform k such that $L_k \subset (Q^{(2)}, f)$ for all $q \in \Lambda$, then we may apply Proposition 5.3 to conclude that $q^{((dk+d^2)n)}$ is in the integral closure of q^n for all n and for all $q \in \Lambda$, where d is the dimension of R. By the Uniform Briançon-Skoda property (see Definition 2.3), we then have that for some uniform t, $q^{((dk+d^2)n)} \subset q^{n-t}$. Using [24, Remark 2.3] will then finish the proof.

The short exact sequence

$$0 \to S/Q \xrightarrow{f} S/Q^{(2)} \to S/(Q^{(2)}, f) \to 0$$

shows that the associated primes P of $S/(Q^{(2)}, f)$ not equal to Q satisfy $\mathrm{depth}(R/q)_P = 1$. Since R/q is normal, all such P have height one over q, and then by assumption, R_{PR} is regular.

In order to prove that $L_k \subset (Q^{(2)}, f)$ for a given k, it suffices to prove this inclusion at all associated primes P of $(Q^{(2)}, f)$. By the paragraph above, all such primes have the property that R_{PR} is regular. But then if we choose k to be the dimension of R_P, which is at most the dimension of R, we have that $(q^{(kn)})_P \subseteq (q_P)^n$ by [14]. Taking $n = 2$ and interpreting this back in S, we have $(L_{2k})_P \subseteq (Q^2 + (f))_P \subseteq (Q^{(2)} + (f))_P$, which is what we want. Finally, if we choose k_0 to be the maximum of the k's obtained from each P associated to $(Q^{(2)}, f)$, as q runs through the elements of Λ, then we have $q^{(k_0 n)} \subseteq q^n$, for all $q \in \Lambda$. □

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